TEMPERATURE DISTRIBUTION IN A CYLINDRICAL CONDUCTOR CARRYING AN ALTERNATING CURRENT

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The temperature distribution in a surface cooled cylindrical conductor heated by a monochromatic current is obtained in the form of a quadrature. The temperature distribution is obtained in final form and is analyzed for weak and strong skin effects.

1. The temperature in a uniform conductor heated by a high-frequency monochromatic current having a distribution described by $\Delta \mathbf{j} = -i\mu\sigma\omega \mathbf{j}$ [1] and cooled at its surface satisfies Poisson's equation

$$\Delta t = -\frac{|j|^2}{2\lambda\sigma}. \tag{1}$$

The t and j distributions were obtained concurrently for a conducting plane layer in [2]. We determine here the temperature distribution in an infinite cylindrical conductor ($\rho \le 1$). If the total current is $\pi r^2 j_0$, $j = j_0 (a\sqrt{i}/2J_1[a\sqrt{i}])J_0(a\rho\sqrt{i})$ [1] and the t distribution is determined by the equation

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dt}{d\rho}\right) = -\frac{(arj_0)^2}{8\lambda\sigma}\left|\frac{J_0(a\rho\sqrt{i})}{J_1(a\sqrt{i})}\right|^2 \tag{2}$$

and the boundary conditions: $t(1) = 0$, $t(0)$ is finite. Integration of (2) using the integral properties of Bessel functions leads to the quadrature of a bilinear combination of Kelvin functions [3].

$$t = \frac{a}{8\lambda\sigma}\left[\frac{rj_0}{|J_1(a\sqrt{i})|}\right]^2 \operatorname{Re}\left[\sqrt{i}\int_\rho^1 J_0(a\rho\sqrt{i})\,J_1(a\rho\sqrt{-i})\,d\rho\right] = \frac{1}{8\lambda\sigma}\left[\frac{rj_0}{|J_1(a\sqrt{i})|}\right]^2 \operatorname{Im}\int_{a\rho\sqrt{i}}^{a\sqrt{i}} J_0(\xi)\,I_1(\xi)\,d\xi. \tag{3}$$

2. For $a \ll 1$ (weak skin effect) Eq. (3) gives approximately

$$t = t_0\left[1 + \frac{a^4}{576}(2\rho^4 + 2\rho^2 - 1)\right], \quad t_0 = \frac{(rj_0)^2}{8\lambda\sigma}(1-\rho^2). \tag{4}$$

The relative difference in the temperatures t and t_0 corresponding to $\omega = 0$ is an increasing function of ρ and is proportional to ω^2 and r^4:

$$\delta = \frac{a^4}{576}(2\rho^4 + 2\rho^2 - 1), \quad \delta(0) = -\frac{a^4}{576}, \quad \delta(1) = \frac{a^4}{192}, \tag{5}$$

which vanishes at $\rho = \sqrt{(\sqrt{3} - 1/2)} \simeq 0.60502$. The average temperature in this approximation is the same for a direct current:

$$\bar{t} \equiv \int_0^1 t(\rho)\,d(\rho^2) = \bar{t}_0 = \frac{1}{2}t_0(0) = \frac{(rj_0)^2}{16\lambda\sigma}. \tag{6}$$

Thus there is a relative "temperature skin effect," a redistribution of temperature with increasing frequency in favor of the periphery of the cross section of the conductor arising from the electromagnetic effect.

3. For $a \gg 1$ (strong skin effect) using the asymptotic formulas for large ξ [3]

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$$J_n(\xi\sqrt{i}) \simeq \frac{1}{\sqrt{2\pi\xi}} \exp\left[\frac{\xi}{\sqrt{2}}(1-i) + i\frac{\pi}{2}\left(n+\frac{1}{4}\right)\right]$$

$$\mathrm{Ei}\,\xi \simeq \frac{\exp\xi}{\xi}, \tag{7}$$

we obtain for the peripheral part of the cross section of the conductor ($a\rho \gg 1$)

$$t \simeq \frac{a}{16\lambda\sigma}(rj_0)^2\left\{1-\frac{1}{\rho}\exp\left[-a(1-\rho)\sqrt{2}\right]\right\}; \tag{8}$$

interpolation in the remaining portion of the cross section gives

$$t(0) \simeq \frac{\pi a}{4\lambda\sigma}(rj_0)^2\exp(-a\sqrt{2})\,\mathrm{Im}\int_0^{a\sqrt{i}} J_0(\xi)\,I_1(\xi)\,d\xi. \tag{9}$$

According to (8), $t(\rho)$ is a decreasing function with the main part of the decrease occurring in a surface layer of thickness $r/a\sqrt{2}$, i.e., half the thickness of the skin layer for j. The rate of fall off increases rapidly in this layer together with a: $t'(1)/t_0'(1) = 2^{-3/2}a^2 \gg 1$. The relative temperature skin effect is very pronounced. The temperature increases with ω and r. The average temperature increases linearly with a:

$$\bar{t} \simeq \frac{1}{2}\bar{t}_0(a-\sqrt{2}) \simeq \frac{a}{2}\bar{t}_0. \tag{10}$$

The relative redistribution of $t(\rho)$ in the transition from a weak to a strong skin effect for an increase in the criterion a (frequency) is somewhat analogous to the evolution of the relative profile (mean local) of the velocity of a liquid in a tube in the transition from laminar to turbulent flow of increasing intensity for an increase of the Reynolds number (flow rate).

NOTATION

λ, σ, and μ are respectively the thermal conductivity, the electrical conductivity, and the absolute magnetic permeability of the conductor;
ω is the angular frequency of the current;
ρ is the distance from the axis of the cylindrical conductor in units of its radius r;
$a = r\sqrt{\mu\sigma\omega}$ is the criterion for the strength of the skin effect;
j is the complex amplitude of the current density;
j_0 is the amplitude of the current density averaged over the cross section of the conductor;
t is the temperature of the conductor measured from the temperature of its surface;
t_0 is the same for a direct current of density $j_0/\sqrt{2}$;
$\delta \equiv (t-t_0)/t_0$.

LITERATURE CITED

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